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LETTER TO THE EDITOR

Solitary waves in a finite depth fluid

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Abstract. It is shown that $u(\xi)=u_0[\cosh^2 a\xi+(ab)^{-2}\sinh^2 a\xi]^{-1}$ is the exact stationary wave solution to the Whitham equation in a two-layer fluid of finite depth and represents the natural connection between the Benjamin–Ono deep water ($a\to 0$) and Korteweg–de Vries shallow water ($b\to\infty$) theories.

The stream function characterising the propagation of a weakly non-linear finite amplitude disturbance in a fluid of total depth D can be written in the form $\psi(x,z,t)=c_0\phi(z)u(x,t)$. The functions ϕ and u are obtained by solution of (Whitham 1967)

$$\frac{d^2\phi}{dz^2}+\left(\frac{N^2(z)}{c^2(k)}-k^2\right)\phi=0 \tag{1}$$

and

$$\begin{gathered}\frac{\partial u(x)}{\partial t}+Cu(x)\frac{\partial u(x)}{\partial x}+\frac{\partial}{\partial x}\int_{-\infty}^{\infty}dx'\,u(x')G(x'-x)=0,\\ G(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,c(k)\,e^{ikx},\end{gathered} \tag{2}$$

subject to the boundary conditions $\phi(0)=\phi(-D)=0$ and $u\to 0$ for $|x|\to\infty$. $N(z)$ is the Brunt–Väisälä frequency, C a parameter characterising the non-linearity and $c(k)$ the phase speed dispersion. Consider a model fluid in which $N^2(z)$ is zero outside of a small range ϵ centred at $z=-d$. For this 'thin thermocline model', solution of equation (1) yields (Phillips 1966)

$$c^2(k)=g\left(\frac{\delta\rho}{\rho_0}\right)[\![k\{k\epsilon+\coth(kd)+\coth[k(D-d)]\}]\!]^{-1}, \tag{3}$$

where $\delta\rho$ denotes the difference in densities between the lower and upper layers and ρ_0 is their mean value. For small enough k-values, neglecting $k\epsilon$, one finds that $c(k)-c_0\sim k^2$ for finite D whereas $c(k)-c_0\sim|k|$ for D infinite. Substitution of these limiting forms into equation (2) yields the Korteweg–de Vries (KDV) (Benjamin 1966) and Benjamin–Ono (BO) (Benjamin 1967, Ono 1975) equations, respectively, corresponding to what are denoted 'shallow' and 'infinitely deep' water theories. Restricting attention to stationary wave solutions, $u(x,t)=u(\xi)$, $\xi=x-ct$, c being the constant wave speed, allows one to directly integrate equation (2) once:

$$cu(\xi)-\tfrac{1}{2}Cu^2(\xi)-\int_{-\infty}^{\infty}d\xi'\,u(\xi')G(\xi'-\xi)=0. \tag{4}$$

The solutions to this equation in the KDV and BO limits are well known:

$$u(\xi)|_{\mathrm{KDV}} = u_0 \operatorname{sech}^2(a\xi), \qquad c - c_0 = \tfrac{1}{3}Cu_0; \tag{5a}$$

$$u(\xi)|_{\mathrm{BO}} = u_0\left[1 + \left(\frac{\xi}{b}\right)^2\right]^{-1}, \qquad c - c_0 = \tfrac{1}{4}Cu_0. \tag{5b}$$

The purpose of the present Letter is to exhibit the comparably simple solution for D finite. For simplicity we assume that $\epsilon \ll d \ll D$, but D finite. For this situation the small $-k$ form of equation (3) becomes

$$c(k) = c_0\left[1 - \tfrac{1}{2}kd\left(\coth(kD) - \frac{1}{kD}\right)\right], \tag{6}$$

where $c_0 = (gd\,\delta\rho/\rho_0)^{1/2}$. We are led in our search for a solution to equation (4) for the $c(k)$ given in equation (6) by the fact that as $D \to \infty$ it should yield for $u(\xi)$ the result given by equation (5b) whereas for $D \to 0$ we expect the $u(\xi)$ of equation (5a). The function $u(\xi)$ which we seek depends on *two* parameters, a and b, and is given by

$$u(\xi; a, b) \equiv \frac{u_0}{\cosh^2(a\xi) + \{[\sinh(a\xi)/a]^2/b^2\}}. \tag{7}$$

When $a \to 0$, b, ξ finite, this function becomes identical with that given in equation (5b) while for $b \to 0$, a, ξ finite, it becomes identical to that given in equation (5a). When $|\xi| \to \infty$, a, b finite, we find

$$u(\xi; a, b) \sim 4u_0[1 + (ab)^{-2}]^{-1}\, e^{-2a|\xi|}, \tag{8}$$

so that if $a \not\equiv 0$, u decays exponentially just as the KDV solution, in contrast to the algebraic decay ($\sim\xi^{-2}$) of the BO solution ($a \equiv 0$).

Although it can be verified that equation (7) represents a solution to equations (2), (4) and (6) by direct substitution, it is easier to show this by first Fourier inverting equation (4) to the form

$$\frac{2}{C}(c - c(k)) = \frac{\int_0^\infty d\xi\, u^2(\xi)\cos(k\xi)}{\int_0^\infty d\xi\, u(\xi)\cos(k\xi)}. \tag{9}$$

To evaluate these integrals we rewrite equation (7) in the equivalent form

$$u(\xi; a, b) = 2B[\cos\delta + \cosh(2a\xi)]^{-1}, \tag{10}$$

with

$$B = u_0P(P+1)^{-1}, \qquad \cos\delta = (P-1)(P+1)^{-1} \tag{11}$$

and $P = (ab)^2$. The required integrals are now elementary and direct substitution of equation (10) into equation (9) gives

$$c(k) = c_0 - BC \operatorname{cosec}\delta\left(\frac{k}{2a}\right)\left[\coth\left(\frac{k\delta}{2a}\right) - \frac{2a}{k\delta}\right] \tag{12}$$

where $c = c_0 - BC \operatorname{cosec}\delta(\cot\delta - \delta^{-1})$. Equations (12) and (6) are identical if we make the identifications

$$\delta = 2aD, \qquad BC = c_0 da \sin\delta, \tag{13}$$

whence verifying that the u given in equation (7) is the desired solution. The parameters u_0, a,b are directly related to the parameters c_0, C, d, D by

$$u_0 b = 2c_0 d/C \qquad a\tan(aD) = b^{-1}. \tag{14}$$

Making use of the result $C = 3c_0/2d$ (Benjamin 1967) we have $u_0 b = 4d^2/3$. The deviation of the wave speed c from c_0 is then given by

$$\frac{c}{c_0} - 1 = \frac{d}{2D}[1 - 2aD\cot(2aD)] = \frac{d}{2D}\left(1 + \frac{D}{b}(1 - a^2b^2)\right). \tag{15}$$

Defining a linewidth Δ for the solution by its full width at its half-peak value, it is given by

$$a\Delta = \cosh^{-1}\left(\frac{1+3(ab)^2}{1+(ab)^2}\right). \tag{16}$$

Finally, the total area under the solution, A, is given by

$$A = 2abu_0D = 8ad^2D/3. \tag{17}$$

If $D \to \infty$, then

$$\frac{c}{c_0} - 1 \cong \frac{d}{2b}\left(1 + \frac{b}{D}\right), \qquad A \cong \tfrac{4}{3}\pi d^2\left(1 - \frac{b}{D}\right), \qquad \text{and} \qquad \Delta \cong 2b\left[1 - \frac{\pi^2}{6}\left(\frac{b}{D}\right)^2\right].$$

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